JOURNAL OF DIFFERENTIAL EQUATIONS **135**, 83–102 (1997)
ARTICLE NO. DE963219

Limits on L^p Regularity of Self-Adjoint Elliptic Operators

E. B. Davies

Department of Mathematics, King's College, Strand, London WC2R 2LS, England

Received July 12, 1996

1. INTRODUCTION

In an earlier paper [10] we have studied uniformly elliptic operators in divergence form with measurable coefficients. These operators act in

$$Hf(x) := \sum_{\substack{|\alpha| \leqslant m \\ |\beta| \leqslant m}} (-1)^{|\alpha|} D^{\alpha}\{a_{\alpha,\beta}(x)\, D^{\beta} f(x)\}, \tag{1}$$

where $a_{\alpha,\beta}(x) = \overline{a_{\beta,\alpha}(x)}$ are complex-valued bounded measurable functions on $\mathbf{R}^N$ for all multi-indices α, β. It is clear that C_c^∞ need not be contained in the domain of such operators. We therefore start from the quadratic form Q defined on C_c^∞ by

$$Q(f) := \int_{\mathbf{R}^N} \sum_{\substack{|\alpha| \leqslant m \\ |\beta| \leqslant m}} a_{\alpha,\beta}(x)\, D^{\beta} f(x)\, \overline{D^{\alpha} f(x)}\, d^N x. \tag{2}$$

The order of any term in the above two equations is defined to be $|\alpha| + |\beta|$. We assume that H is elliptic in the sense that there exist self-adjoint matrices c_1 and c_2 such that the matrix $\{a_{\alpha,\beta}(x)\}_{|\alpha| = |\beta| = m}$ of highest order coefficients satisfies $0 \leqslant c_1 \leqslant a(x) \leqslant c_2$ for all $x \in \mathbf{R}^N$ in the sense of matrices, where the constant coefficient operators

$$C_i := (-1)^m \sum_{|\alpha| = |\beta| = m} c_{i,\alpha,\beta} D^{\alpha+\beta}$$

are uniformly elliptic in the standard sense for $i = 1, 2$. We say that H is homogeneous of order $2m$ if $a_{\alpha,\beta}(x) = 0$ unless $|\alpha| = |\beta| = m$. This ensures that H is non-negative and that $Q(f)$ is comparable in magnitude to $\sum_{|\alpha| = m} \|D^{\alpha} f\|_2^2$. Under the above conditions the closure of the quadratic form Q has domain $W^{m,2}$ and we showed in [10] that the associated semi-bounded self-adjoint operator H has the following further properties.

0022-0396/97 $25.00

If $N<2m$ then the semigroup e^{-Ht} defined on L^2 for $t\geqslant 0$ can be extended to a strongly continuous semigroup on L^p for all $1\leqslant p<\infty$, and the operators have integral kernels $K(t, x, y)$ which satisfy certain generalized Gaussian bounds. However, if $N>2m$ then we were only able to extend the semigroup to L^p for $q_c\leqslant p\leqslant p_c$ where $q_c^{-1}+p_c^{-1}=1$ and

$$p_c:=\frac{2N}{N-2m}.$$

This paper studies two separate problems, linked by a common technique for their solution. The first is to prove that the constants p_c and q_c above are sharp in the sense that for any p outside the stated range there exists a uniformly elliptic operator A for which e^{-At} cannot be extended to a bounded operator on L^p for any $t>0$. The coefficients of the operators which we consider are smooth on $\mathbf{R}^N\backslash\{0\}$ and bounded on $\mathbf{R}^N$.

Let r be a positive integer. The operators of order $4r$ which we consider act on complex-valued functions as in [10], but for those of order $(4r-2)$ we extend the theory of [10] to vector-valued functions, i.e. we consider elliptic systems. This is done as follows. If $\mathscr{H}$ is a finite-dimensional Hilbert space, we let C_c^∞ denote the space of smooth functions of compact support with values in $\mathscr{H}$, and let the coefficients $a_{\alpha,\beta}(x)$ in (1) be operators from $\mathscr{H}$ to $\mathscr{H}$. The equation (2) is then rewritten in the form

$$Q(f):=\int_{\mathscr{H}}\sum_{\substack{|\alpha|\leqslant m\\ |\beta|\leqslant m}}\langle a_{\alpha,\beta}(x)\, D^\beta f(x), D^\alpha f(x)\rangle\, d^N x \tag{3}$$

This could of course be done in matrix language at the cost of greater notational complexity, but the extension to infinite dimensions would then be less clear. It is easy to check that all of the main theorems of [10] carry over to this context with the same proofs, the heat kernel pointwise bounds becoming bounds on the norm of $K(t, x, y)$ as an operator from $\mathscr{H}$ to $\mathscr{H}$.

In Section 2 we consider again some examples discovered independently by de Giorgi [13] and Maz'ya [17] in 1968. We show that these examples yield uniformly elliptic operators of order $2m$ for which an L^p elliptic regularity property fails if p lies outside the range $[q_c, p_c]$. By adapting an argument of Auscher, Coulhon and Tchamitchian [4], who treat only the case $m=1$ and $N\geqslant 5$, we show in Section 3 that if e^{-At} could be extended to a strongly continuous semigroup on L^p this elliptic regularity property would hold. The conclusion is that no such semigroup exists. The dilation invariance of the examples in Section 2 shows that e^{-At} cannot be extended to a bounded operator on L^p for any $t>0$.

In Section 4 we show that the de Giorgi/Maz'ya examples have two distinct peculiarities, one arising from the discontinuity of the coefficients at the origin, and the other from a less obvious singularity at infinity. We first show that any operator whose coefficients have the same limiting behaviour as $x \to 0$ as A gives rise to similar L^p problems for $p > p_c$. We also construct a uniformly elliptic operator with smooth coefficients of the type for which it is known that e^{-Ht} is a strongly continuous semigroup on L^p for all $1 \leqslant p < \infty$. We show that the semigroup is uniformly bounded in L^p norm for p in the interval $[q_c, p_c]$, but that the operator norms may be unbounded as $t \to \infty$ for a larger value of p. We do not obtain any explicit lower bound on the rate of divergence of the norm as $t \to \infty$.

Section 5 deals with the second main problem, whose solution depends upon the same methods. Namely we consider the contraction semigroup $e^{-(H_0+V)t}$ acting on $L^2(\mathbf{R}^N)$ for $t \geqslant 0$, where H_0 is a constant coefficient self-adjoint uniformly elliptic operator of order $2m$ and V is a non-negative smooth potential on $\mathbf{R}^N$. We show that in spite of a large number of positive partial results, it is not true that this semigroup may always be extended to a strongly continuous semigroup on $L^p(\mathbf{R}^N)$ for all $1 < p < \infty$. We review existing results, and show that if $m > 1$ and $N \geqslant 2m+4$ then there exist $p \in (2, \infty)$ and H_0, V such that $e^{-(H_0+V)t}$ cannot be extended to a strongly continuous semigroup acting on $L^p(\mathbf{R}^N)$. This depends upon producing further example of a type first exhibited by Maz'ya and Nazarov [18]. Namely for such values of m, N there exists a constant coefficient homogeneous operator H_0 of order $2m$ whose Green function is non-positive (by which we mean negative on a non-empty set). It seems to be difficult to find a useful characterization of the symbols of those constant coefficient elliptic operators with this property. The symbol of H_0 is convex, but if we give up this condition we are able to construct a constant coefficient elliptic operator H_0 with non-positive Green function for all $N \geqslant 2m+3$, a result which appears to be new even for the case $2m = 4$ and $N = 7$.

2. THE EXAMPLES OF DE GIORGI/MAZ'YA

We start by considering the second order example. This acts on suitably regular functions $f: \mathbf{R}^N \to \mathbf{C}^N$ where $N \geqslant 3$ according to the formula

$$A_2 f(x) := -\Delta f(x) + B^* B f(x),$$

where B takes $\mathbf{C}^N$-valued functions to complex-valued functions according to the formula

$$Bf(x) := \sum_{\alpha, i=1}^{N} \left\{ \mu \delta_{i,\alpha} + \nu \frac{x_i x_\alpha}{|x|^2} \right\} \frac{\partial f_i}{\partial x_\alpha}$$

for constants μ and ν to be chosen below. We may also write A_2 in the form

$$(A_2 f)_i = -\sum_{j,\alpha,\beta} \frac{\partial}{\partial x_\alpha}\left\{a^{i,j}_{\alpha,\beta}(x)\frac{\partial f_j}{\partial x_\beta}\right\}.$$

We note that the coefficients $a^{i,j}_{\alpha,\beta}(x)$ are smooth and bounded for $x \neq 0$.

Let $O(N)$ denote the group of orthogonal $N\times N$ matrices, and consider its unitary actions on $L^2(\mathbf{R}^N, \mathbf{C}^N)$ defined for $U \in O(N)$ by

$$V_U f(x) := U\{f(U^{-1}x)\}$$

and on $L^2(\mathbf{R}^N, \mathbf{C})$ by

$$W_U f(x) := f(U^{-1}x).$$

It may be verified that the V-invariant subspace $\mathscr{L}$ of $L^2(\mathbf{R}^N, \mathbf{C}^N)$ is

$$\mathscr{L} := \left\{f(x) = \frac{x}{|x|}\, g(x)\colon g \in L^2(\mathbf{R}^N, \mathbf{C}) \text{ is a radial function}\right\}.$$

Lemma 1. *The self-adjoint operator A_2 acting in $L^2(\mathbf{R}^N, \mathbf{C}^N)$ commutes with the group representation $\{V_U\colon U \in O(N)\}$ in the sense of spectral projections, so the subspace $\mathscr{L}$ is invariant under A_2 and its spectral projections.*

Proof. It may be verified directly that

$$BV_U f = W_U Bf$$

for all $f \in C_c^\infty$ and $U \in O(N)$. This implies that $Q(V_U f) = Q(f)$ for all $f \in C_c^\infty$ and then for all $f \in W^{m,2}$ by approximation. The procedure for constructing A_2 from its quadratic form then leads to the desired conclusions.

We now make the choices $\mu := \lambda(N-2)$ and $\nu := N\lambda$, where $\lambda > 0$ is to be determined. A direct calculation shows that the function $f_i(x) := x_i/|x|^\gamma$ satisfies $A_2 f(x) = 0$ for all $x \neq 0$ provided

$$\gamma(N-\gamma) - \lambda^2(N-1)^2\,(N-2\gamma)^2 = 0. \tag{4}$$

For f to lie in $W^{1,2}_{\mathrm{loc}}$ we also need $\gamma < N/2$. One of the two solutions of (4) is

$$\gamma = \frac{N}{2}\left\{1 - \{4\lambda^2(N-1)^2 + 1\}^{1/2}\right\}. \tag{5}$$

We thus have:

LEMMA 2. *If* $0 < 2\delta < N-2$ *then there exists* $\lambda > 0$ *such that the unbounded vector-valued function* $f_i(x) := x_i/|x|^{\delta+1}$ *lies in* $W^{1,2}_{\mathrm{loc}}$ *and satisfies* $A_2 f(x) = 0$ *for all* $x \neq 0$.

The condition $N > 2$ is needed to ensure that $W^{1,2}(\{x\colon |x| < 1\}) = W^{1,2}(\{x\colon 0 < |x| < 1\})$. The existence of a suitable λ for any δ in the stated range follows from (5).

Given any integer $s \geqslant 1$ we now consider the operator A_{4s+2} acting on $\mathbf{C}^N$-valued functions in $L^2(\mathbf{R}^N)$ according to the formula

$$A_{4s+2} := (-\Delta)^s A_2 (-\Delta)^s.$$

More precisely A_{4s+2} is the uniformly elliptic non-negative self-adjoint operator of order $(4s+2)$ associated with the closed form

$$Q(f) := \|A_1^{1/2}(-\Delta)^s f\|^2$$

with domain $W^{(2s+1),2}(\mathbf{R}^N)$.

LEMMA 3. *If* $0 < 2\sigma < N-(4s+2)$ *there exists* $\lambda > 0$ *such that the function* $f_i(x) := x_i/|x|^{\sigma+1}$ *lies in* $W^{(2s+1),2}_{\mathrm{loc}}$ *and satisfies* $A_{4s+2} f(x) = 0$ *for all* $x \neq 0$.

Proof. The first two conditions of the lemma are needed to ensure that $f \in W^{(2s+1),2}_{\mathrm{loc}}$. If we put $\delta := \sigma + 2s$ then a direct calculation shows that $(-\Delta)^s f(x) = c x_i/|x|^{\delta+1}$ where $c \neq 0$, and the proof can be completed by an application of Lemma 1.

We now turn to operators of order $4t$ for some integer $t \geqslant 1$. These are slightly simpler in that they act on scalar-valued functions.

LEMMA 4. *If* $0 < 2\tau < N - 4t$ *there exists a uniformly elliptic operator* A_{4t+4} *such that the function* $f(x) := |x|^{-\tau}$ *lies in* $W^{(2t+2),2}_{\mathrm{loc}}$ *and satisfies* $A_{4t+4} f(x) = 0$ *for all* $x \neq 0$.

Proof. For $t = 1$ the procedure is to define

$$A_4 := -\nabla \cdot A_2 \nabla$$

by the quadratic form method, the domain of the quadratic form being $W^{2,2}(\mathbf{R}^N)$. The lemma is then a corollary of Lemma 2. If $t > 1$ we put

$$A_{4t+4} := (-\Delta)^t A_4 (-\Delta)^t$$

and proceed as in Lemma 3.

THEOREM 5. *Let m be a positive integer and let $0<2\mu<N-2m$. Then there exists a uniformly elliptic operator $A=A_{N,2m,\mu}$ of order $2m$ acting in $L^2(\mathbf{R}^N)$ (with vector or scalar values depending on m) and a function $g\in\mathrm{Dom}(A)$ such that g is of compact support, $g(x)$ is smooth for $x\neq 0$ and satisfies $|g(x)|\sim|x|^{-\mu}$ as $x\to 0$, and such that $Ag\in C_c^\infty(\mathbf{R}^N)$.*

Proof. We start with the function f defined by one of the above lemmas, depending upon the value of m. The required function g is then of the form $g(x):=f(x)\,\phi(x)$ where $\phi\in C_c^\infty(\mathbf{R}^N,\mathbf{R})$ equals 1 is some neighbourhood of the origin. The fact that $g\in\mathrm{Dom}(A)$ follows from $g\in\mathrm{Dom}(A^{1/2})=W^{m,2}$ and $Ag\in L^2$ by a standard argument [9, Theorem 1.2.7].

For the rest of the paper we write A for the operators constructed in Theorem 5, where N, m, μ are any constants satisfying $0<2\mu<N-2m$. We also write $a_{\alpha,\beta}(x)=a_{N,2m,\mu,\alpha,\beta}(x)$ for their (possibly matrix-valued) coefficients. These satisfy

$$a_{\alpha,\beta}(sx)=a_{\alpha,\beta}(x) \tag{6}$$

for all $x\in\mathbf{R}^N$ and $s>0$, and are bounded smooth functions of x on $\mathbf{R}^N\backslash\{0\}$.

3. THE SEMIGROUPS ASSOCIATED WITH A

In this section we study the one-parameter semigroups e^{-At} defined on $L^2(\mathbf{R}^N)$ for $t\geqslant 0$ using the spectral theorem. We reformulate some theorems of Auscher, Coulhon and Tchamitchian [4, 6, 7] concerning extensions to L^p of a strongly continuous semigroup e^{-Ht} acting on $L^2(X,\mathscr{H},dx)$, where H is a non-negative self-adjoint operator, and all functions take their values in an auxiliary Hilbert space $\mathscr{H}$. We assume that H has the following two properties.

(i) For some p_1 such that $2<p_1<\infty$ there exists a constant M such that

$$\|e^{-Ht}f\|_{p_1}\leqslant M\,\|f\|_{p_1}$$

for all $f\in L^2\cap Lp^1$ and $t>0$.

(ii) There exist $p_2>2$ and $c<\infty$ such that

$$\|f\|_{p_2}\leqslant c(\|H^{1/2}f\|_2+\|f\|_2)$$

for all $f\in\mathrm{Dom}(H^{1/2})$.

LEMMA 6. *Assuming* (i), *there exists a strongly continuous uniformly bounded semigroup* $T_p(t)$ *acting on* L^p *for all* $2 \leqslant p \leqslant p_1$ *such that*

$$T_p(t) f = e^{-Ht} f$$

for all $f \in L^2 \cap L^p$ *and all* $t \geqslant 0$. *The generator* $-H_p$ *of* $T_p(t)$ *is consistent with* $-H$ *in the sense that*

$$(H_p + \lambda)^{-1} f = (H + \lambda)^{-1} f$$

for all $\lambda > 0$ *and* $f \in L^2 \cap L^p$.

Proof. The existence of a uniformly bounded semigroup on L^p for $2 \leqslant p \leqslant p_1$ follows by interpolation, and the only issue is its strong continuity. If q is the index conjugate to p and if $f \in L^2 \cap L^p$, $g \in L^2 \cap L^q$ then by the strong continuity of e^{-Ht} in L^2

$$\lim_{t \to 0} \langle e^{-Ht} f, g \rangle = \langle f, g \rangle.$$

A density argument now implies that

$$\langle T_p(t) f, g \rangle = \langle f, g \rangle$$

for all $f \in L^p$ and $g \in L^q$. Since $(L^p)^* = L^q$ the first part of the proof is completed by using [8, Prop. 1.23]. The statement concerning the consistency of the resolvents is a consequence of the standard formula

$$(H + \lambda)^{-1} f = \int_0^\infty e^{-\lambda t} e^{-Ht} f \, dt$$

valid for all $f \in L^2 \cap L^p$, and convergent in the L^2 and L^p norms.

Since the semigroups and resolvents are consistent for different p we henceforth simply refer to them as e^{-Ht} and $(H + \lambda)^{-1}$ respectively. The following lemma is a slight modification of a result of Coulhon [6, 7], and has essentially the same proof.

LEMMA 7. *Assume conditions* (i) *and* (ii) *and let* $2 \leqslant b \leqslant c \leqslant p_1$ *Then*

$$\| e^{-Ht} f \|_c \leqslant k t^{-a} \| f \|_b$$

for all $0 < t \leqslant 1$ *and* $f \in L^b$, *where*

$$a := (b^{-1} - c^{-1}) \frac{p_2}{p_2 - 2}.$$

COROLLARY 8. *There exist a finite sequence* $2=u_0<u_1<\cdots<u_k=p_1$ *and constants* $c_1, \ldots, c_k$ *such that*

$$\|(H+1)^{-1} f_i\|_{u_i} \leqslant c_i \|f_i\|_{u_{i-1}}$$

for all $1\leqslant i\leqslant k$ *and* $f_i \in L^{u_{i-1}}$.

Proof. By estimating the integral in the formula

$$(H+1)^{-1} f=\int_0^\infty e^{-t} e^{-Ht} f\, dt$$

we obtain

$$\|(H+1)^{-1} f\|_c \leqslant k \|f\|_b$$

whenever

$$b^{-1}-c^{-1}<1-\frac{2}{p_2}.$$

This result is not sharp, but it yields the statement of the corollary by subdividing $[2, p_1]$ into a large enough number of subintervals of equal length.

THEOREM 9. [4] *Under the conditions* (i) *and* (ii) *if* $f\in \mathrm{Dom}(H)\subseteq L^2$ *and* $Hf\in L^2\cap L^{p_1}$ *then* $f\in \mathrm{Dom}(H_{p_1})\subseteq L^{p_1}$.

Proof. If $f\in \mathrm{Dom}(H)$ we put $g:=(H+1)f\in L^2$ and deduce from Corollary 8 that $f\in L^{u_1}$. This implies $g\in L^{u_1}$ and hence $f\in L^{u_2}$. An inductive argument yields $f\in L^{p_1}$. Finally the identity $f=(H+1)^{-1} g$ where $g\in L^{p_1}$ implies $f\in \mathrm{Dom}(H_{p_1})$.

We now return to the particular operators A. The following theorem generalises the main result of [4] to systems and to the case $m>1$, but simplifies the proof even if $m=1$. Our main interest in it is that it identifies the precise range of values of p for which such examples can exist.

THEOREM 10. *Let* $N>2m$. *The semigroup* e^{-At} *can be extended from* $L^2\cap L^p$ *to uniformly bounded strongly continuous semigroup on* L^p *provided* $q_c\leqslant p\leqslant p_c$. *If* p *does not lie in this range then for a suitable choice of the parameter* $\mu>0$ *the operator* e^{-At} *cannot be extended from* $L^2\cap L^p$ *to a bounded operator on* L^p *for any* $t>0$.

Proof. The first statement is taken from [10, Theorem 25], except that the uniform boundedness follows by a scaling argument of the type described in [5, Lemma 6]; see below. Now suppose that $N > 2m$ and that $p > p_c$. Choose the constant μ of Theorem 5 close enough to $(N-2m)/2$ so that the function g of that theorem does not lie in L^p. Since

$$\mathrm{Dom}(H^{1/2}) = W^{m,2}(\mathbf{R}^N)$$

the condition (ii) is satisfied by A with $p_2 = 2N/(N-2m)$ by a standard Sobolev embedding theorem.

If S_s is the scaling operator

$$(S_s f)(x) = f(sx) \tag{7}$$

where $s > 0$ and $x \in \mathbf{R}^N$, then

$$\|S_s f\|_p = s^{-N/p} \, \|f\|_p \tag{8}$$

for all $f \in L^p$ and $1 \leqslant p \leqslant \infty$. Since A only has coefficients of order $2m$ and these satisfy (6), it may be seen that

$$S_s^{-1} A S_s = s^{2m} A.$$

This implies that the L^p norm of e^{-At} is independent of $t > 0$. If $p = p_1$ and this norm is finite for any $t > 0$ then condition (i) is satisfied, and the conclusion of Theorem 9 follows. This contradicts Theorem 5.

Finally if $p < q_c$ the statement of the theorem follows by a duality argument.

4. SOME FURTHER EXAMPLES

In our next two theorems we show that the peculiar L^p behaviour of $A = A_{N,2m,\mu}$ arises both from the discontinuity of the coefficients at $x = 0$ and from a less obvious singularity at infinity. Our results depend heavily upon the scaling property (6) of the coefficients $a_{\alpha,\beta}(x)$ of A.

We now consider another homogeneous uniformly elliptic operator B of order $2m$ acting in $L^2(\mathbf{R}^N)$ whose coefficients $b_{\alpha,\beta}(x)$ are bounded. We define B rigorously as a self-adjoint operator by the quadratic form method.

THEOREM 11. *Assume in addition to the above conditions that*

$$\lim_{s \to 0} b_{\alpha,\beta}(sx) = a_{\alpha,\beta}(x)$$

for all α, β and $x \in \mathbf{R}^N$. If $q_c \leqslant p \leqslant p_c$ then the operators e^{-Bt} are uniformly bounded in L^p norm for $0 \leqslant t < \infty$. If, however, $p > p_c$ and μ is large enough then the operators e^{-Bt} cannot be extended from $L^2 \cap L^p$ to provide a strongly continuous semigroup on L^p.

Proof. The case $q_c \leqslant p \leqslant p_c$ is treated as in Theorem 10. Now let $p > p_c$ and let μ be large enough that e^{-At} is an unbounded operator on L^p for all $t > 0$. We shall derive a contradiction from the assumption that $\|e^{-Bt}\|_{p,p} \leqslant c$ for all $0 \leqslant t \leqslant 1$.

Defining S_s by (7) a direct calculation shows that the operator $B_s := s^{2m} S_s B S_s^{-1}$ is of the form

$$B_s f(x) := \sum_{|\alpha| = |\beta| = m} (-1)^m D^\alpha\{b_{\alpha,\beta}(sx)\, D^\beta f(x)\}.$$

The computations are all carried out for the quadratic forms on their common domain $W^{m,2}$. Since the coefficients of B_s converge to those of A, Proposition 13 below implies that

$$\lim_{s \to 0} \langle e^{-B_s} f, g\rangle = \langle e^{-A} f, g\rangle$$

for all $f \in L^2 \cap L^p$ and all $g \in L^2 \cap L^q$, where q is the index conjugate to p. It also follows from (8) that

$$\|e^{-B_s}\|_{p,p} = \|e^{-B_s^{2m}}\|_{p,p} \leqslant c$$

for all $s \in [0, 1]$. A routine density argument now implies that $\|e^{-A}\|_{p,p} \leqslant c$, which is the required contradiction.

Our next example is more surprising in that uniformly elliptic operators with smooth coefficients are in most respects extremely well behaved [1, 16, 20]. Indeed for many purposes uniform continuity of the highest order coefficients is sufficient to guarantee good L^p behaviour for all p [2, 3, 19].

THEOREM 12. *Assume that the coefficients of the operator B are smooth and bounded together with all of their derivatives. Assume also that*

$$\lim_{s \to \infty} b_{\alpha,\beta}(sx) = a_{\alpha,\beta}(x)$$

for all α, β and $x \in \mathbf{R}^N$. If $q_c \leqslant p \leqslant p_c$ then the semigroup e^{-Bt} is uniformly bounded in L^p norm as $t \to \infty$. If, however, $p > p_c$ and μ is large enough then

$$\lim_{t \to \infty} \|e^{-Bt}\|_{p,p} = +\infty.$$

Proof. It is known [16, 20] that the first condition of the theorem implies that e^{-Bt} extends to a strongly continuous semigroup $T_p(t)$ on L^p for all $1 \leqslant p < \infty$. The proof of this theorem is an obvious modification of that of the previous one, and again uses Proposition 13.

This proposition is not new, but we have not been able to find a precise reference in the literature.

PROPOSITION 13. *Let H_n be homogeneous uniformly elliptic operators of order 2m acting in $L^2(\mathbf{R}^N)$ of the form*

$$H_n f(x) := \sum_{\substack{|\alpha| \leqslant m \\ |\beta| \leqslant m}} (-1)^{|\alpha|} D^\alpha \{a_{n,\alpha,\beta}(x) D^\beta f(x)\}, \tag{9}$$

where $0 \leqslant c_1 \leqslant a_n(x) \leqslant c_2$ for all n and $x \in \mathbf{R}^N$ in the sense of matrices, and where c_i are as in Section 1. Suppose that the operator H has a similar expression and that

$$\lim_{n \to \infty} a_{n,\alpha,\beta}(x) = a_{\alpha,a}(x)$$

for all α, β and $x \in \mathbf{R}^N$. Then H_n converge to H in the strong resolvent sense.

Proof. There exist two sequences $b_n(x)$ and $b'_n(x)$ of matrix-valued coefficients such that

$$c_1 \leqslant b_n(x) \leqslant a_n(x) \leqslant b'_n(x) \leqslant c_2$$

for all n and x in the sense of matrices, and such that $b_n(x)$ increase monotonically to $a(x)$ while $b'_n(x)$ decrease monotonically to $a(x)$ as $n \to \infty$. Let B_n and B'_n be the corresponding self-adjoint operators. Then $B_n \leqslant H_n \leqslant B'_n$ and

$$(B'_n + 1)^{-1} \leqslant (H_n + 1)^{-1} \leqslant (B_n + 1)^{-1}$$

by [8, Theorem 4.17]. The convergence of the coefficients implies monotone convergence of the quadratic forms in the sense of [8, Theorem 4.32] and hence implies that $(B_n + 1)^{-1}$ decreases monotonically to $(H+1)^{-1}$ while $(B'_n + 1)^{-1}$ increases monotonically to $(H+1)^{-1}$ as $n \to \infty$. If $C_n := (H_n + 1)^{-1} - (B'_n + 1)^{-1}$ then $C_n \geqslant 0$ and C_n converge weakly to 0. This implies that C_n converge strongly to 0, and then that $(H_n + 1)^{-1}$ converge strongly to $(H+1)^{-1}$.

5. GENERALIZED SCHRÖDINGER SEMIGROUPS

In this section we consider self-adjoint operators defined as quadratic form sums by $H := H_0 + V$ acting in $L^2(\mathbf{R}^N)$ where $H_0 \geqslant 0$ is a constant coefficient elliptic self-adjoint operator which is homogeneous of order $2m$

and V is a non-negative smooth potential. The quadratic form domain of H is thus

$$\mathrm{Dom}(H^{1/2}) = W^{m,2}(\mathbf{R}^N) \cap \{f \in L^2 : V^{1/2} f \in L^2\}.$$

It may be verified that C_c^∞ is a quadratic form core for H. The Trotter product formula implies that e^{-Ht} is a one-parameter contraction semigroup on L^2 for $t \geqslant 0$. The formula cannot be applied for $p \neq 2$ because $e^{-H_0 t}$ may not be a contraction semigroup on L^p for such p. Our goal in this section is to show that this is not just a technical objection. In spite of a number of partial positive results, listed below, there exist N, m, p, H_0, V such that e^{-Ht} cannot be extended to a strongly continuous one-parameter semigroup on L^p. We disprove the following hypothesis, formulated for a particular choice of N, m, p, H_0:

(H) For all non-negative smooth potentials V on $\mathbf{R}^N$, the contraction semigroup $e^{-(H_0+V)t}$ on $L^2(\mathbf{R}^N)$ can be extended to a strongly continuous semigroup on $L^p(\mathbf{R}^N)$.

For $m=1$ this hypothesis is valid for all choices of N, p, H_0 by the Trotter product or Feynmann-Kac formulas [15, 21, 8, Section 4.5]. For $N<2m$ it is valid for all p, H_0 by [5]. For $N>2m$ it is valid for H_0 and $q_c \leqslant p \leqslant p_c$ by [10]. For bounded potentials it is valid for all N, m, p, H_0 by a standard perturbation argument [8, Theorem 3.1], and one may even extend this result to all potentials in a suitably defined Kato class [15]. The discrete analogue of (H) is valid for all N, m, p, H_0 because in this case H_0 is a bounded perturbation of the potential V. The only negative indication is that for time-dependent potentials the corresponding result can be false even for $N=1$ and $m=2$ by [11].

Our counterexample to (H) does not describe the full picture and we conjecture that there exists H_0 for which it is false whenever $N>2m$ and $p>p_c$, and also that the validity of the hypothesis depends upon the choice of H_0, i.e. that some constant coefficient elliptic operators have quite different behaviour from others for the type of question we are considering. The proof of the falsity of (H) involves a chain of lemmas leading to a contradiction of the hypothesis.

Lemma 14. *If* (H) *is valid then there exist constants $c_t < \infty$ such that*

$$\|e^{-(H_0+V)t}\|_{p,p} \leqslant c_t$$

for all $t>0$ and all non-negative smooth V.

Proof. Given $t>0$ suppose that for all positive integers n there exist $V_n \geqslant 0$ such that $\|e^{-(H_0+V_n)t}\|_{p,p} \geqslant n$. We construct a non-negative smooth potential V such that $\|e^{-(H_0+V_n)t}\|_{p,p}$ is not finite, contradicting the hypothesis (H).

Let $B_{m,n}$ be balls with centres $c_{m,n}$ and radii $2m$ which are disjoint for all positive integers $m, n \geqslant 1$. Let $0 \leqslant f_{m,n} \in C_c^\infty(\mathbf{R}^N)$ satisfy

$$f_{m,n}(x) = \begin{cases} V_n(x) & \text{if} \quad |x| \leqslant m \\ 0 & \text{if} \quad |x| \geqslant 2m. \end{cases}$$

Then define

$$V(x) := \sum_{m,n=1}^{\infty} f_{m,n}(x - c_{m,n}).$$

Clearly $0 \leqslant V \in C^\infty$, since in each ball $B_{m,n}$ at most one term of this series is non-zero.

If the translation operator T_a on L^p is defined for $a \in \mathbf{R}^N$ by

$$T_a f(x) := f(x-a)$$

then

$$T_{c_{m,n}}(H_0 + V)\, T_{-c_{m,n}} = H_0 + W_{m,n}$$

where $W_{m,n}(x) = V_n(x)$ if $|x| \leqslant m$. By sandwiching $\{W_{m,n}\}_{m=1}^{\infty}$ between two monotonically converging sequences of potentials both with the pointwise limit V_n, one may show as in Proposition 13 that $H_0 + W_{m,n}$ converges in the strong resolvent sense to $H_0 + V_n$ as $m \to \infty$. Using the fact that T_a are L^p isometries, we finally deduce that

$$\begin{aligned} n \leqslant \|e^{-(H_0+V_n)t}\|_{p,p} &\leqslant \liminf_{m \to \infty} \|e^{-(H_0+W_{m,n})t}\|_{p,p} \\ &= \liminf_{m \to \infty} \|T_{-c_{m,n}} e^{-(H_0+V)t} T_{c_{m,n}}\|_{p,p} \\ &= \|e^{-(H_0+V)t}\|_{p,p} \end{aligned}$$

for all integers n.

Lemma 15. *If* (H) *is valid then there exists a constant $c < \infty$ such that*

$$\|e^{-(H_0+V)t}\|_{p,p} \leqslant c$$

for all $t > 0$ and all non-negative smooth V.

Proof. Let c be the constant c_t of Lemma 14 for $t=1$. If S_s is defined by (7) then

$$S_s^{-1}(H_0+V)\,S_s = s^{2m}(H_0+V_s)$$

where

$$V_s(x) := s^{2m}V(s^{-1}x).$$

If $t>0$ and we put $s:=t^{-1/(2m)}$ then we obtain

$$\begin{aligned}\|e^{-(H_0+V)\,t}\|_{p,\,p} &= \|e^{-S_s(H_0+V_s)\,S_s^{-1}}\|_{p,\,p}\\ &= \|S_s e^{-(H_0+V_s)}S_s^{-1}\|_{p,\,p}\\ &= \|e^{-(H_0+V_s)}\|_{p,\,p}\\ &\leqslant c\end{aligned}$$

as required.

THEOREM 16. *Let Ω be any region in $\mathbf{R}^N$, let W be a non-negative smooth potential on $\mathbf{R}^N$, and let $H_\Omega := H_0+W$ act on $L^2(\Omega)$ subject to Dirichlet boundary conditions. If* (H) *is valid then $\|e^{-H_\Omega t}\|_{p,\,p}\leqslant c$ for all $t>0$. Moreover if $f\in \mathrm{Dom}(H_\Omega)$ and $H_\Omega f\in L^2\cap L^p$ then $f\in L^p$.*

Proof. The proof uses the theory of monotone limits of closed quadratic forms whose domains need not be dense subspaces, as described in [8, Chapter 4]. Let Ω_n be a sequence of bounded subregions of Ω with smooth boundaries, such that $\Omega_n\subseteq\overline{\Omega_n}\subseteq\Omega$ for all n and $\bigcup_{n=1}^{\infty}\Omega_n=\Omega$. Let V_n be non-negative smooth potentials on $\mathbf{R}^N$ such that $V_n(x)=0$ if and only if $x\in\overline{\Omega_n}$. The quadratic forms associated with the operators H_0+W+mV_n increase monotonically as $m\to+\infty$, the limits being the closed forms

$$Q_n(f) := \|H_0^{1/2}f\|_2^2+\|W^{1/2}f\|_2^2$$

whose domains are

$$\{f\in W^{m,\,2}(\mathbf{R}^N)\colon W^{1/2}f\in L^2 \text{ and } \operatorname{supp} f\subseteq\overline{\Omega_n}\}.$$

These forms decrease monotonically as $n\to+\infty$, the domain of the limit Q_∞ being

$$\{f\in W_0^{m,\,2}(\Omega)\colon W^{1/2}f\in L^2\}.$$

The operator H_Ω is (by definition) the self-adjoint operator associated with Q_∞.

Given (H) we deduce from Lemma 15 that

$$\|e^{-Ht}\|_{p,p} \leqslant \liminf_{n\to\infty} \liminf_{m\to\infty} \|e^{-(H_0+W+mV_n)t}\|_{p,p} \leqslant c$$

for all $t \geqslant 0$. The final step is an application of Theorem 9.

COROLLARY 17. *If $N \geqslant 8$ then there exists H_0 of order 4 and $p \in (2, \infty)$ for which* (H) *is not valid.*

Proof. Theorem 1 of Maz'ya and Nazarov [18] constructs a constant coefficient elliptic operator H_0 of order 4, a conical region $K \subseteq \mathbf{R}^N$ with vertex at the origin, and an unbounded function $u \in W^{2,2}_{\mathrm{loc}}(\bar{K})$ with $u = \partial u/\partial n = 0$ on ∂K, such that $f := H_0 u \in C_0^\infty(\bar{K})$. If ϕ is a radial function in $C_c^\infty(\mathbf{R}^N)$ which equals 1 in some neighbourhood of the origin then $f := u\phi$ lies in $W_0^{2,2}(K)$ and $H_0 f \in C_0^\infty(\bar{K})$, so $f \in \mathrm{Dom}(H_0)$. Theorem 1 of [18] only asserts that u is unbounded, but the construction actually establishes that $f \notin L^p$ for large enough p.

The proof of [18] depends crucially upon the construction of an explicit operator H_0 of order 4 for which the Green function of H_0^{-1} is not positive if $N \geqslant 8$. In the remainder of this paper we construct an operator of homogeneous order $2m$ with the same property whenever $m > 1$ and $N \geqslant 2m+4$. A feature of this operator is that its symbol is not only elliptic but convex. We conjecture that the condition $N \geqslant 2m+4$ cannot be weakened for such symbols. It is well known [14, Theorem 7.1.20] that the Green function G of such an operator is a homogeneous function of degree $2m - N$ and C^∞ on $\mathbf{R}^N \backslash \{0\}$, and the only problem is to determine its sign at a selected point.

Let r, s be two positive integers such that $r + s = N$ and let H act in $L^2(\mathbf{R}^N)$ by the formula

$$H := (-\Delta_{r,x})^m + (-\Delta_{s,y})^m \tag{10}$$

where $-\Delta_{r,x}$ denotes the Laplacian acting in $\mathbf{R}^r$ and the variable is $x \in \mathbf{R}^r$. Then

$$e^{-Ht} f = K_t * f$$

for all $f \in L^p$, $1 \leqslant p < \infty$, where

$$K_t(x, y) := k_{r,t}(x)\, k_{s,t}(y)$$

and

$$k_{r,t}(x) := \frac{1}{(2\pi)^r} \int_{\mathbf{R}^r} e^{ix\cdot\xi - |\xi|^{2m} t}\, d\xi.$$

The Green function (or fundamental solution) is

$$G(x, y) := \int_0^\infty K_t(x, y)\, dt.$$

THEOREM 18. *If* $s := 2m + 3$ *and* $r := N - s \geqslant 1$ *then the Green function satisfies* $G(x, 0) < 0$ *for all* $x \in R^r$.

Proof. Because the operator H is homogeneous we have

$$k_{r,t}(x) = t^{-r/2m} k_{r,1}(xt^{-1/2m}).$$

Hence

$$G(x, 0) = C_1 \int_0^\infty t^{-N/2m} k_{r,1}(xt^{-1/2m})\, dt.$$

Here and below c_i refer to positive constants depending on m, r, s, which could be computed explicitly if desired. Since $k_{r,1}(x)$ is invariant under rotations, a change of variables leads to the expression

$$G(x, 0) = c_2\, |x|^{2m-N} \int_{\mathbf{R}^r} |u|^{s-2m}\, k_{r,1}(u)\, du.$$

We now put $s := 2m + 3$ and $r := N - s \geqslant 1$ to obtain the result of the theorem provided $I(r, 3) < 0$, where

$$I(r, \gamma) := \int_{\mathbf{R}^r} |u|^\gamma\, k_{r,1}(u)\, du. \tag{11}$$

Note that since $m > 1$ is assumed to be an integer the function $k_{r,1}$ lies in Schwartz space and the integral (11) is absolutely convergent for all $\operatorname{Re} \gamma > -r$. The proof that $I(r, 3) < 0$ is given below, using a direct computation.

PROPOSITION 19. *If* $r \geqslant 1$ *then* $I(r, 3) < 0$.

Proof. We start with the case $r \geqslant 2$, the argument for $r = 1$ being slightly more difficult. Using the fact that $k_{r,1}$ lies in Schwartz space, we have

$$I(r, 3) = \int_{\mathbf{R}^r} |u|^{-1}\, |u|^4\, k_{r,1}(u)\, du$$

$$= c_3 \int_{\mathbf{R}^r} |\xi|^{1-r}\, \Delta_\xi^2\, e^{-|\xi|^{2m}}\, d\xi. \tag{12}$$

We evaluate the derivatives in polar coordinates to obtain

$$\Delta_\xi^2 e^{-|\xi|^{2m}} = (2ma\,|\xi|^{2m-4} + 4m^2 b\,|\xi|^{4m-4} + 8m^3 c\,|\xi|^{6m-4} + 16m^4 d\,|\xi|^{8m-4})\, e^{-|\xi|^{2m}}$$

where

$$\begin{aligned} a &:= -(r+2m-2)(2m-2)(r+2m-4) \\ b &:= (r+2m-2)(r+6m-6) + (4m-2)(r+4m-4) \\ &= 28m^2 + 12mr - 48m + r^2 - 10r + 20 \\ c &:= -(2r+12m-8) \\ d &:= 1. \end{aligned}$$

We evaluate the integral in (12) using

$$\int_0^\infty v^{\beta-4} e^{-v^{2m}}\, dv = \frac{1}{2m}\,\Gamma\left(\frac{\beta-3}{2m}\right).$$

The end result is

$$I(r, 3) = c_5\,\Gamma\left(1 - \frac{3}{2m}\right)(1 - r^2)$$

which is negative if $r \geqslant 2$ and $m \geqslant 2$.

The proof for $r = 1$ is similar except that we evaluate $I(1, 3+\varepsilon)$ for $0 < \varepsilon < 1$. The limit $\varepsilon \to 0+$ is taken at the very end of the computation, using standard properties of the Gamma function.

Note 1. The algebraic calculations in the last two propositions were done by hand and also checked using computer algebra.

Note 2. The integral $I(r, \gamma)$ can also be evaluated by using the fact that it is analytic function of γ for $\operatorname{Re}\gamma > -r$. If $-r < \gamma < 0$ then the evaluation is a straightforward exercise involving the Fourier transform of the tempered distribution $x \to |x|^\gamma$; see [14, Exercise 7.1.35, p. 411]. The result is an expression involving Gamma functions which can be continued to $\gamma = 3$ and then evaluated, yielding a negative value.

Note 3. An examination of the proof shows that Theorem 18 is still valid for $H := (-\Delta_{r,x})^m + H_{s,m}$ where $H_{s,m}$ is any constant coefficient homogeneous elliptic operator of order $2m$ acting in $L^2(\mathbf{R}^s)$ and $s = 2m+3$.

Note 4. The operator H defined by (10) may also be studied for non-integral m by the same method provided $2 < 2m < N$. However the choice $s := 2m + 3$ is no longer appropriate and the results are much more complicated to express.

If we do not require the symbol of the operator to be convex, then it is possible to produce examples with non-positive Green functions in lower space dimensions. We consider an operator H_0 acting in $L^2(\mathbf{R}^N)$ where $N = r + s$. We let a variable in $\mathbf{R}^N$ be denoted by (x, y) or (ξ, η) where $x, \xi \in \mathbf{R}^r$ and $y, \eta \in \mathbf{R}^s$. We suppose that H_0 is elliptic of homogeneous order $2m > 2$ and that its symbol is of the form

$$P(\xi, \eta) := \sum_{i=0}^{m} |\xi|^{2(m-i)} Q_i(\eta),$$

where Q_i are homogeneous polynomials of degree $2i$ and $Q_0 = 1$.

THEOREM 20. *If $N > 2m + 3$ then there exists an elliptic operator H_0 with constant coefficients of the above form for which the Green function is not everywhere positive.*

Proof. The fundamental solution of the heat equation is

$$K_t(x, y) := \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} \exp[ix \cdot \xi + iy \cdot \eta - P(\xi, \eta)\, t]\, d^r\xi\, d^s\eta.$$

Therefore

$$K_t(x, 0) = t^{-N/2m} \check{F}(t^{-1/2m} x),$$

where

$$\check{F}(x) := \frac{1}{(2\pi)^N} \int_{\mathbf{R}^r} F(\xi)\, e^{ix \cdot \xi}\, d^r\xi$$

and

$$F(\xi) := \int_{\mathbf{R}^s} e^{-P(\xi, \eta)}\, d^s\eta.$$

It follows from our assumptions that F and $\check{F}$ are both rotationally invariant functions in Schwartz space. If $N > 2m$ then the Green function satisfies

$$
\begin{aligned}
G(x, 0) &:= \int_0^\infty K_t(x, 0)\, dt \\
&= \int_0^\infty t^{-N+2m} \check{F}(t^{1/2m} x)\, dt \\
&= c_1\, |x|^{2m-N} \int_{\mathbf{R}^r} |u|^{N-2m-r}\, \hat{F}(u)\, d^r u,
\end{aligned}
$$

where c_i denote various positive constants. We now choose $r := N - 2m - 2$ or equivalently $s := 2m + 2$ to obtain

$$
\begin{aligned}
G(x, 0) &= c_1\, |x|^{2m-N} \int_{\mathbf{R}^r} |u|^2\, \check{F}(u)\, d^r u \\
&= -c_2\, |x|^{2m-N}\, \Delta_\xi F(0)
\end{aligned}
$$

so the proof is complete provided it is possible to have $\Delta_\xi F(0) > 0$.

It follows from the expansion

$$
e^{-P(\xi, \eta)} = e^{-Q_m(\eta)}(1 - |\xi|^2\, Q_{m-1}(\eta) + O(|\xi|^4))
$$

and standard estimates of the remainder term that

$$
\Delta_\xi f(0) = -2r \int_{\mathbf{R}^s} e^{-Q_m(\eta)} Q_{m-1}(\eta)\, d^s \eta
$$

so we only require the integral to be negative. A typical example which satisfies all of the conditions is

$$
P(\xi, \eta) := |\xi|^{2m} - a|\xi|^2\, |\eta|^{2m-2} + |\eta|^{2m},
$$

where $m > 1$ and $a > 0$ is small enough to maintain the ellipticity of H_0. The conditions $s = 2m + 2$ and $r \geqslant 1$ yield $N \geqslant 2m + 3$.

ACKNOWLEDGMENTS

I thank A. M. Hinz for many discussions at an early stage of this work and V. G. Maz'ya for some important communications.

REFERENCES

1. S. Agmon, "Lectures on Elliptic Boundary Problems," Van Nostrand, Princeton, NJ, 1965.

2. S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, *Commun. Pure Appl. Math.* **12** (1959), 623–727.
3. S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, II, *Commun. Pure Appl. Math.* **17** (1964), 35–92.
4. P. Auscher, T. Coulhon, and P. Tchamitchian, Absence de principe du maximum pour certaines équations paraboliques complexes, *Coll. Math.*, to appear.
5. G. Barbatis and E. B. Davies, Sharp bounds on heat kernels of higher order elliptic operators, *J. Operator Theory* **36** (1996), 179–198.
6. T. Coulhon, Dimension à l'infini d'un semi-groupe analytique, *Bull. Sci. Math.* **114**, No. 2 (1990), 485–500.
7. T. Coulhon, Inégalités de Gagliardo–Nirenberg pour les semi-groupes d'opérateurs et applications, *Potential Anal.* **1** (1992), 343–353.
8. E. B. Davies, "One-Parameter Semigroups," Academic Press, San Diego, 1980.
9. E. B. Davies, "Heat Kernels and Spectral Theory," Cambridge Univ. Press, Cambridge, UK, 1989.
10. E. B. Davies, Uniformly elliptic operators with measurable coefficients, *J. Functional Anal.* **132** (1995), 141–169.
11. E. B. Davies, Long time asymptotics of fourth order parabolic equations, *J. Anal. Math.* **67** (1995), 323–345.
12. E. B. Davies and A. M. Hinz, Kato class potentials for higher order elliptic operators, *J. London Math. Soc.*, to appear.
13. E. de Giorgi, Un esempio di estemali discontinue per un problema variazionale di tipo ellitico, *Boll. Un. Mat. Ital.* **1**, No. 4 (1968), 135–137.
14. L. Hörmander, "The Analysis of Linear Partial Differential Operators I," 2nd ed., Springer-Verlag, Berlin, 1989.
15. T. Kato, Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups, *in* "Topics in Functional Analysis" (I. Gohberg and M. Kac, Eds.), pp. 185–195, Academic Press, San Diego, 1978.
16. Yu. A. Kordyukov, L^p-theory of elliptic differential operators on manifolds of bounded geometry, *Acta Applic. Math.* **23** (1991), 223–260.
17. V. G. Maz'ya, Examples of nonregular solutions of quasilinear elliptic equations with analytic coefficients, *Functional Anal. Appl.* **2**, No. 3 (1968), 230–234.
18. V. G. Maz'ya and S. A. Nazarov, The vertex of a cone can be nonregular in the Wiener sense for a fourth order elliptic operator, *Math. Notes* **39** (1986), 14–16; Transl. of *Mat. Zametki* **39**, (1986) 24–28.
19. C. B. Morrey, Jr, "Multiple Integrals in the Calculus of Variations," Springer-Verlag, Berlin, 1966.
20. D. W. Robinson, "Elliptic Operators and Lie Groups," Oxford Univ. Press, London, 1991.
21. B. Simon, "Functional Integration and Quantum Physics," Academic Press, New York, 1979.